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UNSYMMETRIC LANCZOS MODEL REDUCTION AND LINEAR STATE FUNCTION OBSERVER FOR FLEXIBLE STRUCTURES

by

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Unsymmetric Lanczos Model Reduction and Linear State Function Observer for Flexible Structures

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ABSTRACT

This report summarizes part of the research work accomplished during the second year of a two-year grant. The research, entitled "Application of Lanczos Vectors to Control Design of Flexible Structures" concerns various ways to use Lanczos vectors and Krylov vectors to obtain reduced-order mathematical models for use in the dynamic response analyses and in control design studies. This report presents a one-sided, unsymmetric block Lanczos algorithm for model reduction of structural dynamics systems with unsymmetric damping matrix, and a control design procedure based on the theory of linear state function observers to design low-order controllers for flexible structures.

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Chapter 1

INTRODUCTION

This report summarizes part of the research work accomplished during the second year of a two-year grant on the topic of the application of Krylov vectors and Lanczos vectors to the control of flexible structures. The goal of this research project is to develop a reduced-order modeling technique for general linear systems based on the use of Krylov vectors and Lanczos vectors, and to apply the technique to the modeling of flexible structures. Some advantages of the Krylov-based and Lanczos-based reduced-models in the control design application are also cited. Accomplishments made during the first year of this project include the development of a Krylov model-reduction algorithm for structural dynamics systems, the formulation of a substructure-based control design procedure called the Substructural Controller Synthesis (SCS) method, and the development of a controller-reduction method based on the preservation of impulse response energy. These methods are summarized in the first year's interim report, Ref. [6]

This report includes another two topics concerned with model reduction and controller design of flexible structures. In Chapter 2, a one-sided, unsymmetric block Lanczos algorithm for structural dynamics systems with unsymmetric damping matrix is derived. Although there are several existing unsymmetric Lanczos algorithms [3, 10, 19], the one-sided, unsymmetric block

Lanczos algorithm proposed in this report has the following advantages: (1) the numerical breakdown problem that usually occurs in the two-sided unsymmetric Lanczos method is not present, (2) the Lanczos vectors that are created lie in the controllable and observable subspace, (3) the reduced-order model is guaranteed to be stable, (4) a shifting scheme can be used for unstable systems, and (5) the flexibility of the choice of starting vector leads to more accurate reduced-order models.

In Chapter 3 of this report, a control design procedure based on the linear state function observer is described. This method is a semi-inverse design procedure in that the control law is not designed before the observer system, but is a result that comes from the observer design. However, the observer design is not completely independent of the control design either, but seeks to yield a feedback signal that is close to a prescribed control law. First, the observer design problem is considered as the reconstruction of a linear function of the state vector. The linear state function to be reconstructed is the given control law. Then, based on the theory for linear state function observers, the observer design is formulated as a parameter optimization problem. The optimization objective is to generate a matrix that is close to the given feedback gain matrix. Based on that matrix, the form of the observer and a new control law can be determined. The semi-inverse design procedure can yield a reduced-order observer with dimension considerably smaller than that of the system.

Chapter 2 and Chapter 3 of this report are essentially revised versions of technical papers that are already published or to be published. The readers may find some symbols used in Chapter 2 are re-used in Chapter 3 but with completely different meanings. To avoid confusion, the readers are advised to

consider Chapter 2 and Chapter 3 as two independent topics with their own notations.

Chapter 2

AN UNSYMMETRIC LANCZOS MODEL REDUCTION ALGORITHM

The Lanczos algorithm is a three-term iteration scheme originally developed by Lanczos [12] as a method for evaluating the eigensolution of symmetric matrices. The set of vectors generated by the Lanczos algorithm forms a basis for the *Krylov subspace*. When used as a transformation basis, the set of Lanczos vectors transforms a matrix to a tridiagonal form, from which the eigenvalues can be determined easily. Recently, there has been a lot of research concerning the application of Lanczos vectors and Krylov vectors to the dynamic analysis and control design of structural dynamics systems. For undamped structural dynamics systems, the Lanczos model reduction method was first proposed by Nour-Omid and Clough in Refs. [17, 18]. Later, Craig and Hale applied Krylov vectors to model reduction in the context of coupled substructures [5]. Recently, Su and Craig incorporated the concept of parameter-matching model reduction with Krylov vectors to develop a Krylov-model reduction method for dynamic analysis and control of flexible structures [22, 24]. The Krylov vectors and the Lanczos vectors basically span the same subspace, because they are generated by the same iteration formula. The only difference is that the Lanczos algorithm employs a special three-term orthogonalization scheme. A damped Krylov model-reduction algorithm is also presented in Ref. [22] for the model reduction of damped structural dynamics

systems described by a second-order matrix differential equation. This was the first Krylov vector method to include the damping matrix in the algorithm. However, there is a restriction on the method in Ref. [22]: the damping matrix has to be symmetric.

Although most passive damping mechanisms yield a symmetric damping matrix, there are cases when the damping matrix is unsymmetric. For structures, unsymmetric damping may arise from a combination of active feedback control design and Coriolis forces. To deal with general, nonsymmetric damping, the usual approach is to write the system's dynamic equation in the first-order state-space form. Then, an unsymmetric Lanczos algorithm is used to create a basis for model reduction. In Refs [10] and [11], Kim and Craig present an unsymmetric block Lanczos algorithm that generates a set of *left Lanczos vectors* and a set of *right Lanczos vectors*. These two set of Lanczos vectors form a basis that transforms the system equation to an unsymmetric tridiagonal form. Reference [19] also has a similar two-sided unsymmetric Lanczos iteration scheme for general, non-classically damped systems. For systems with symmetric damping matrix and symmetric stiffness matrix, Ref. [19] further shows that the two-sided unsymmetric Lanczos algorithm can be simplified into a one-sided algorithm by taking advantage of symmetry. The major disadvantage of a two-sided Lanczos algorithm is that the reduced-order model obtained may exhibit some high frequency modes or even unstable modes, although the full-order system is stable. Other applications of Lanczos vectors and Krylov vectors include the skew-symmetric Lanczos algorithm of Gupta and Lawson for spinning structures [8] and the Krylov-vector controller-reduction method of Su and Craig [25].

In this chapter, another unsymmetric Lanczos algorithm for structural dynamics systems with unsymmetric damping matrix and/or unsymmetric stiffness matrix is proposed. The advantages of the present method over the other unsymmetric Lanczos algorithms are: (1) the numerical breakdown problem that usually occurs in applying the two-sided unsymmetric Lanczos method is not present, (2) the Lanczos vectors that are produced lie in the controllable and observable subspace, (3) the reduced-order model is guaranteed to be stable, (4) a shifting scheme can be used for unstable systems, and (5) the flexibility of the choice of starting vector leads to more accurate reduced-order models.

Detailed development of the algorithm is presented in Section 2.1. Modification of the algorithm for the application to different cases is discussed in Section 2.2. Finally, Section 2.3 presents numerical examples to illustrate the efficiency of the proposed algorithm. Part of the material in this chapter has been submitted as a conference paper (Ref. [23]).

2.1 Development of an Unsymmetric Block Lanczos Iteration Scheme

2.1.1 Transformation of the System Equation

Consider a linear, time-invariant system described by

$$\begin{aligned} \dot{x} &= Az + Bu & x \in R^n, u \in R^l \\ y &= Cx & y \in R^m \end{aligned} \tag{2.1}$$

Assume that the system is stable and completely controllable. Then, the following Lyapunov equation has a unique positive definite solution.

$$AW_c + W_c A^T + BB^T = 0 \quad (2.2)$$

W_c is called the *controllability grammian* of the system. If the system's state vector is transformed to another set of coordinates through a projection matrix L

$$x = L\bar{x} \quad (2.3)$$

then the system equation becomes

$$\begin{aligned} \dot{\bar{x}} &= \bar{A}\bar{x} + \bar{B}u \\ y &= \bar{C}\bar{x} \end{aligned} \quad (2.4)$$

where the system matrices in the new coordinates are

$$\bar{A} = L^{-1}AL \quad , \quad \bar{B} = L^{-1}B \quad , \quad \bar{C} = CL \quad (2.5)$$

The controllability grammian of the system in the new coordinates satisfies

$$\bar{A}\bar{W}_c + \bar{W}_c\bar{A}^T + \bar{B}\bar{B}^T = 0$$

or

$$L^{-1}AL\bar{W}_c + \bar{W}_c(L^{-1}AL)^T + L^{-1}BB^TL^{-T} = 0 \quad (2.6)$$

From Eq. (2.2) and Eq. (2.6), we obtain the following relationship

$$\bar{W}_c = L^{-1}W_cL^{-T} \quad \text{or} \quad \bar{W}_c^{-1} = L^TW_c^{-1}L \quad (2.7)$$

which indicates that the controllability grammian is not invariant under coordinate transformation.

Now, assume the transformation is such that the controllability grammian in the new coordinates is an identity matrix. Then,

$$\bar{W}_c = I \longrightarrow L^T W_c^{-1} L = I \longrightarrow L^{-1} = L^T W_c^{-1} \quad (2.8)$$

In this case, the transformed system matrices can be expressed by

$$\bar{A} = L^T W_c^{-1} A L, \quad \bar{B} = L^T W_c^{-1} B, \quad \bar{C} = C L \quad (2.9)$$

If it is further assumed that the projection matrix L is such that the \bar{B} matrix has the special form

$$\bar{B} = \begin{bmatrix} B_1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad (2.10)$$

then the Lyapunov equation for the controllability grammian, Eq. (2.6), becomes

$$\bar{A} + \bar{A}^T + \begin{bmatrix} B_1 B_1^T & 0 & \dots & 0 \\ 0 & 0 & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{bmatrix} = 0 \quad (2.11)$$

which indicates that the new system matrix \bar{A} is almost skew-symmetric, except for the first diagonal block, which is symmetric. (A skew-symmetric matrix N satisfies $N + N^T = 0$.) In summary, for any stable linear, time-invariant system, if the projection matrix L is such that in the new coordinates the controllability grammian is an identity matrix and the transformed input matrix \bar{B} has nonzero entries only in the first block, then the transformed system matrix \bar{A} is almost skew-symmetric matrix. The purpose of showing this transformation is to prepare for the development of a one-sided, unsymmetric Lanczos algorithm in the next section. The derivation is basically inspired by the form of the Lyapunov equation in Eq. (2.11) and the Lanczos iteration scheme of Ref. [10].

2.1.2 The Lanczos Iteration Algorithm

In this section, a one-sided, three-term, block-Lanczos iteration scheme will be developed. This iteration scheme, together with a special choice of starting block of vectors, can be used to create a set of Lanczos vectors. The projection matrix formed by the set of Lanczos vectors transforms the system matrix A into an almost skew-symmetric, block-tridiagonal form.

First, assume that there exists a set of blocks of vectors $Q_i, i = 1, 2, \dots, k$, that satisfy

$$A[Q_1 \ Q_2 \ \cdots \ Q_k] = [Q_1 \ Q_2 \ \cdots \ Q_k] \begin{bmatrix} \mathcal{F}_1 & \mathcal{G}_1 & & & \\ \mathcal{H}_1 & \mathcal{F}_2 & \mathcal{G}_2 & & \\ & \mathcal{H}_2 & \mathcal{F}_3 & \cdot & \\ & & \cdot & \cdot & \cdot \\ & & & \cdot & \cdot & \cdot \\ & & & & \cdot & \cdot \end{bmatrix} \quad (2.12)$$

The above equation implies the following iteration formula

$$AQ_i = Q_{i-1}\mathcal{G}_{i-1} + Q_i\mathcal{F}_i + Q_{i+1}\mathcal{H}_i \quad (2.13)$$

Let us further assume that the Q_i 's are orthogonalized with respect to the inverse of the controllability grammian. That is

$$Q_i^T W_c^{-1} Q_j = \begin{cases} I & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases} \quad (2.14)$$

Then, by premultiplying Eq. (2.13) by $Q_{i-1}^T W_c^{-1}$, $Q_i^T W_c^{-1}$, and $Q_{i+1}^T W_c^{-1}$ respectively, and using the orthogonality condition in Eq. (2.14), it can be shown that

$$\begin{aligned} \mathcal{G}_{i-1} &= Q_{i-1}^T W_c^{-1} A Q_i \\ \mathcal{F}_i &= Q_i^T W_c^{-1} A Q_i \\ \mathcal{H}_i &= Q_{i+1}^T W_c^{-1} A Q_i \end{aligned} \quad (2.15)$$

Let the starting block of vectors of the iteration formula in Eq. (2.13) be the B matrix after being normalized with respect to W_c^{-1} . That is, let

$$Q_1 = BU\Sigma^{-\frac{1}{2}} \quad \text{and} \quad U\Sigma U^T = B^T W_c^{-1} B \quad (2.16)$$

where $U\Sigma U^T$ is the singular-value decomposition of $B^T W_c^{-1} B$ with $U^T U = I$ and Σ being the diagonal matrix containing the singular values. With this special choice of starting block of vectors, two identities can be derived. The first identity is,

$$\begin{aligned} \mathcal{G}_i + \mathcal{H}_i^T &= Q_i^T W_c^{-1} A Q_{i+1} + (Q_{i+1}^T W_c^{-1} A Q_i)^T \\ &= Q_i^T (W_c^{-1} A + A^T W_c^{-1}) Q_{i+1} \end{aligned}$$

Combining Eqs. (2.2) and (2.16) with the above equation gives

$$\begin{aligned} \mathcal{G}_i + \mathcal{H}_i^T &= -Q_i^T (W_c^{-1} B B^T W_c^{-1}) Q_{i+1} \\ &= -Q_i^T [W_c^{-1} (Q_1 \Sigma^{\frac{1}{2}} U^{-1}) (Q_1 \Sigma^{\frac{1}{2}} U^{-1})^T W_c^{-1}] Q_{i+1} = 0 \end{aligned}$$

or,

$$\mathcal{H}_i = -\mathcal{G}_i^T \quad (2.17)$$

The second identity is

$$\begin{aligned} \mathcal{F}_i + \mathcal{F}_i^T &= Q_i^T W_c^{-1} A Q_i + Q_i^T A^T W_c^{-1} Q_i \\ &= -Q_i^T (W_c^{-1} B B^T W_c^{-1}) Q_i \\ &= -Q_i^T [W_c^{-1} (Q_1 \Sigma^{\frac{1}{2}} U^{-1}) (Q_1 \Sigma^{\frac{1}{2}} U^{-1})^T W_c^{-1}] Q_i \end{aligned}$$

or,

$$\mathcal{F}_i + \mathcal{F}_i^T = \begin{cases} -\Sigma & \text{if } i = 1 \\ 0 & \text{if } i \neq 1 \end{cases} \quad (2.18)$$

The two identities in Eqs. (2.17) and (2.18) indicate that the block tridiagonal matrix in Eq. (2.12) is almost skew-symmetric, except the diagonal elements of \mathcal{F}_1 . Therefore, if we define

$$L \equiv \begin{bmatrix} Q_1 & Q_2 & \cdots & Q_k \end{bmatrix} \quad (2.19)$$

and

$$T \equiv \begin{bmatrix} \mathcal{F}_1 & \mathcal{G}_1 & & & \\ -\mathcal{G}_1^T & \mathcal{F}_2 & \mathcal{G}_2 & & \\ & -\mathcal{G}_2^T & \mathcal{F}_3 & \cdot & \\ & & & \cdot & \cdot & \\ & & & & \cdot & \cdot & \cdot \end{bmatrix} \quad (2.20)$$

then, Eq. (2.12) can be written as

$$AL = LT \quad (2.21)$$

Applying the orthogonality condition in Eq. (2.14) to the above equation gives

$$\bar{A} \equiv L^T W_c^{-1} AL = T \quad (2.22)$$

In summary, the unsymmetric Lanczos algorithm proposed in this chapter is described by the following three-term iteration formula

$$AQ_i = Q_{i-1}\mathcal{G}_{i-1} + Q_i\mathcal{F}_i - Q_{i+1}\mathcal{G}_i^T \quad (2.23)$$

The starting blocks of vectors are $Q_0 = 0$ and $Q_1 = BU\Sigma^{-\frac{1}{2}}$. At the i -th iteration, the new block of vectors Q_{i+1} is to be determined by this iteration formula, with Q_j , $j = 1, 2, \dots, i$, being already obtained in the previous iterations. Let

$$R_i = -Q_{i+1}\mathcal{G}_i^T$$

or, from Eq. (2.23),

$$R_i \doteq AQ_i - Q_{i-1}\mathcal{G}_{i-1} - Q_i\mathcal{F}_i \quad (2.24)$$

in which \mathcal{F}_i and \mathcal{G}_{i-1} are determined by using the formulas in Eqs. (2.15a) and (2.15b). Then, the new block of vectors Q_{i+1} can be obtained by simply normalizing R_i with respect to W_c^{-1} . So, let the singular-value decomposition of $R_i^T W_c^{-1} R_i$ be

$$R_i^T W_c^{-1} R_i = U \Sigma U^T = U_\alpha \Sigma_\alpha U_\alpha^T \quad (2.25)$$

where U_α and Σ_α are the non-singular portions. Then, the new block of vectors can be calculated by

$$Q_{i+1} = R_i U_\alpha \Sigma_\alpha^{-\frac{1}{2}} \quad (2.26)$$

The vectors generated by Eq. (2.24) have an interesting property. That is

$$\text{span} \left[Q_1, Q_2, \dots, Q_k \right] = \text{span} \left[Q_1, AQ_1, \dots, A^{k-1}Q_1 \right]$$

The set of vectors on the right-hand side of the above equation is called a set of *Krylov vectors*, which can be generated by the simple iteration formula

$$Q_{i+1} = AQ_i \quad \text{with} \quad Q_1 \text{ given}$$

The difference between the Krylov vectors and the Lanczos vectors is that the latter are orthogonalized. Therefore, the Lanczos vectors and the Krylov vectors span the same subspace. Since the B matrix is used as the starting block of vectors ($Q_1 = BU\Sigma^{-\frac{1}{2}}$), the Lanczos vectors created lie in the span of $\left[B, AB, \dots, A^{k-1}B \right]$, which is the *controllability matrix* of the system. Thus, the projection matrix L defined in Eq. (2.19) has a column subspace that is the same as that of the system's controllability matrix.

2.1.3 Model Reduction

The set of Lanczos vectors generated by the Lanczos iteration formula can be used as a basis for model reduction. Let the L matrix be partitioned into

$$L = \begin{bmatrix} L_R & L_T \end{bmatrix}$$

where $L_R \in R^{n \times r}$ corresponds to the retained portion and $L_T \in R^{n \times (n-r)}$ corresponds to the truncated portion. Then, the state vector can be decomposed into

$$x = L_R x_R + L_T x_T \quad (2.27)$$

The reduced system equation is described by

$$\begin{aligned} \dot{x}_R &= T_R x_R + \bar{B}_R u \\ y &= \bar{C}_R x_R \end{aligned} \quad (2.28)$$

where

$$T_R = L_R^T W_c^{-1} A L_R, \quad \bar{B}_R = L_R^T W_c^{-1} B, \quad \bar{C}_R = C L_R \quad (2.29)$$

Since the matrices of the reduced system satisfy

$$T_R + T_R^T + \bar{B}_R \bar{B}_R^T = 0$$

the controllability grammian of the reduced system is an identity matrix. Also, as will be explained later, the reduced system in Eq. (2.28) is a completely controllable system. According to the Lyapunov theorem [2], if a controllable system has a positive definite controllability grammian, then the system is stable. Therefore, the reduced system obtained by the Lanczos model reduction method is a stable system.

Table 2.1: The one-sided, unsymmetric block-Lanczos algorithm.

Given A , B , and C .
(1) Solve $AW_c + W_cA^T + BB^T = 0$ for W_c .
(2) Starting vectors:
(a) $Q_0 = 0$
(b) $R_0 = B$
(c) $R_0^T W_c^{-1} R_0 = U \Sigma U^T = U_\alpha \Sigma_\alpha U_\alpha^T$ (singular-value decomposition)
(d) $Q_1 = R_0 U_\alpha \Sigma_\alpha^{-\frac{1}{2}}$ (normalization)
(3) For $i = 1, 2, \dots$ repeat:
(e) $R_i = A Q_i - Q_{i-1} \mathcal{G}_{i-1} - Q_i \mathcal{F}_i$ (orthogonalization)
$\mathcal{G}_{i-1} = Q_{i-1}^T W_c^{-1} A Q_i$, $\mathcal{F}_i = Q_i^T W_c^{-1} A Q_i$
(f) $R_i^T W_c^{-1} R_i = U \Sigma U^T = U_\alpha \Sigma_\alpha U_\alpha^T$ (singular-value decomposition)
(g) $Q_{i+1} = R_i U_\alpha \Sigma_\alpha^{-\frac{1}{2}}$ (normalization)
end
(4) Form the k -block transformation matrix $L = [Q_1 \ Q_2 \ \dots]$.

2.2 Modification and Application of the Proposed Lanczos Algorithm

The Lanczos iteration scheme developed in the previous section is summarized by the algorithm shown in Table 2.1. This algorithm assumes that the system considered is stable and completely controllable. For application to systems that are not stable and/or not completely controllable, slight modifications of the algorithm must be made. Also, for model-reduction applications, it can be shown that a different choice of starting vectors yields reduced-order

models with different properties. Also, there is an alternative approach, which uses the observability grammian as the normalization weighting matrix. Details about modifications of the algorithm for different applications are discussed in this section.

2.2.1 A Shifting Method for Unstable Systems

In Step (1) of the algorithm, the controllability grammian, which is to be used as the normalization weighting matrix, must be obtained by solving the Lyapunov equation. However, the Lyapunov equation has a unique semi positive-definite solution if and only if the system matrix A is stable. For unstable systems, one can use a simple shifting approach to ensure a unique solution of the Lyapunov equation, and then apply the Lanczos algorithm to the shifted system.

First, let

$$A_\sigma = A - \sigma I \quad (2.30)$$

where σ is chosen such that all of the eigenvalues of A_σ lie in the left half of the complex plane. Then, since A_σ is stable, the controllability grammian for the shifted system can be obtained from

$$A_\sigma W_{\sigma c} + W_{\sigma c} A_\sigma^T + BB^T = 0 \quad (2.31)$$

After the controllability grammian $W_{\sigma c}$ is calculated, the proposed Lanczos iteration scheme can be used to generate a set of Lanczos vectors for the shifted system. The only change in the algorithm is to use A_σ as the iteration matrix and to use $W_{\sigma c}^{-1}$ as the normalization weighting matrix. The L matrix formed

by the generated Lanczos vectors satisfies

$$L^T W_{\sigma_c}^{-1} L = I \quad , \quad L^T W_{\sigma_c}^{-1} A_\sigma L = T \quad (2.32)$$

By using the above identities, the original system matrix can be transformed to

$$\bar{A} \equiv L^T W_{\sigma_c}^{-1} A L = L^T W_{\sigma_c}^{-1} (A_\sigma + \sigma I) L = T + \sigma I \quad (2.33)$$

which is in block-tridiagonal form.

The subspace spanned by the shifted Lanczos vectors is

$$\text{span} \left[B, (A - \sigma I)B, \dots (A - \sigma I)^{k-1}B \right]$$

which is the same as the span of the controllability matrix of the unshifted system, that is, the span of $\left[B, AB, \dots A^{k-1}B \right]$. Therefore, the controllability property of the Lanczos subspace is not altered by the shifting.

2.2.2 The Pseudo-Inverse of the Controllability Grammian

In the proposed Lanczos algorithm, the inverse of the controllability grammian, W_c^{-1} , is required for orthonormalization (Steps (c) and (f), singular-value decomposition). For systems that are not completely controllable, the controllability grammian is semi positive-definite, which means that it is singular and that its inverse does not exist. In this case, a pseudo-inverse approach must be used instead.

Since W_c is symmetric, its singular-value decomposition takes the form

$$W_c = \begin{bmatrix} \Phi_c & \Phi_\varepsilon \end{bmatrix} \begin{bmatrix} \Lambda_c & \\ & 0 \end{bmatrix} \begin{bmatrix} \Phi_c^T \\ \Phi_\varepsilon^T \end{bmatrix} \quad (2.34)$$

In the above decomposition, the column vectors of Φ_c form the controllable subspace and the column vectors of $\Phi_{\bar{c}}$ form the uncontrollable subspace. The pseudo-inverse of W_c can be defined as

$$W_c^+ = \Phi_c \Lambda_c^{-1} \Phi_c^T \quad (2.35)$$

which corresponds to the controllable (or nonsingular) portion of the system. Then, W_c^+ instead of W_c^{-1} can be used in Steps (c) and (f) to perform normalization.

By orthonormalizing R_i with respect to the inverse (or pseudo-inverse) of the controllability grammian, the uncontrollable part is removed from the Lanczos vectors. This can be explained by using the fact that $R_i^T W_c^+ R_i = R_i^T \Phi_c \Lambda_c^{-1} \Phi_c^T R_i$ is singular if and only if some column vectors of R_i are linearly-dependent and/or are orthogonal to Φ_c . Therefore, to retain only the non-singular portion of $R_i^T W_c^+ R_i$ at the normalization step, Steps (f) and (g), is the same as to retain only the linearly-independent and controllable portion of R_i . As a result, the Lanczos vectors generated by the proposed algorithm are assured to lie in the span of the controllable subspace.

2.2.3 Starting Vectors

If the objective is simply to transform a matrix into an almost skew-symmetric, block-tridiagonal form like the T matrix in Eq. (2.20), then the choice of the starting block of vectors is completely arbitrary. One can choose an arbitrary starting block of vectors Q_1 , solve for X from the Lyapunov equation $AX + XA^T + Q_1 Q_1^T = 0$ and, then use itsnverse (or pseudo-inverse if X is

singular) as the normalization weighting matrix in the Lanczos iteration process. The projection matrix L formed by the generated Lanczos vectors will transform the matrix A into an almost skew-symmetric form. However, for model reduction purposes, the choice of starting vectors is not a trivial issue.

As mentioned previously, by choosing B as the starting block of vectors, the span of the L matrix formed by the Lanczos vectors is the same as the span of $\begin{bmatrix} B, & AB, & \dots & A^{k-1}B \end{bmatrix}$. As a result, the reduced system matrices in Eq. (2.9) satisfy

$$\bar{C}\bar{A}^i\bar{B} = (CL)(L^TW_c^{-1}AL)^i(L^TW_c^{-1}B) = CA^iB \quad (2.36)$$

for $i = 0, 1, \dots, k-1$. The recursive proof procedure employed in Ref. [25] can be used to prove the property in Eq. (2.36). The parameters CA^iB , $i = 0, 1, \dots$, are called the *Markov parameters* of the system. Therefore, by using Lanczos vectors as the basis for model reduction, the reduced system preserves a certain number of Markov parameters of the full-order system. By matching Markov parameters, the reduced system tends to approximate the high frequency range of the original system.

Besides the Markov parameters, another set of system parameters that are important is $CA^{-i}B$, $i = 1, 2, \dots$, which are called the *low-frequency moments* [25]. For some applications, it might be important to preserve some of the system's low-frequency moments as well as some Markov parameters. It can be shown that if the projection matrix L is such that

$$\text{span}\{L\} = \text{span}\left[A^{-j}B, \dots, A^{-1}B, B, AB, \dots, A^k B\right] \quad (2.37)$$

with $j, k > 0$, then the reduced system matches the system parameters CA^iB , for $i = -j, \dots, 0, \dots, k$.

Obviously, if $A^{-j}B$ is chosen as the starting block of vectors for the iteration formula in Eq. (2.23), then the set of Lanczos vectors generated lie in the same span as the matrix in Eq. (2.37). Therefore, one can simply assume that the system considered is

$$\begin{aligned} \dot{x} &= Ax + \hat{B}u \\ y &= Cx \end{aligned} \tag{2.38}$$

where

$$A^j \hat{B} = B$$

or

$$\hat{B} = A^{-j}B$$

and, then use $A^{-j}B$ as the starting block of vectors, and use the inverse (or pseudo-inverse) of the solution of

$$AW_c + W_c A^T + A^{-j}B(A^{-j}B)^T = 0$$

as the normalization weighting matrix for the proposed Lanczos iteration algorithm. The set of Lanczos vectors thus generated will form a projection matrix L such that Eq. (2.37) is satisfied and the transformed system matrix has an almost skew-symmetric, block tridiagonal form. The \bar{B} matrix, however, no longer has the form in Eq. (2.10), but contains nonzero elements up to $(j+1)$ -th block. That is

$$\bar{B}^T = (L^T W_c^{-1} B)^T = \begin{bmatrix} \mathcal{B}_1^T, & \cdots & \mathcal{B}_{j+1}^T, & 0, & \cdots & 0 \end{bmatrix}^T$$

2.2.4 The Use of A^{-1} as the Iteration Matrix

If A^{-1} instead of A is used as the iteration matrix in Eq. (2.23), which becomes

$$A^{-1}Q_i = Q_{i-1}\mathcal{G}_{i-1} + Q_i\mathcal{F}_i - Q_{i+1}\mathcal{G}_i^T \quad (2.39)$$

then the set of Lanczos vectors generated will lie in the span of the generalized controllability matrix $\begin{bmatrix} A^{-1}B, & A^{-2}B, & \dots & A^{-k}B \end{bmatrix}$. Then the resulting reduced system matches a certain number of low-frequency moments, and thus, approximates the low frequency range of the full-order system. For structural dynamics systems, the low frequency range usually is the dominant frequency range and, therefore, should be well-approximated by the reduced-order model. In this case, the iteration formula in Eq. (2.39) is recommended.

The system in Eq. (2.1) can be rewritten as

$$\begin{aligned} A^{-1}\dot{x} &= x + A^{-1}Bu \\ y &= Cx \end{aligned} \quad (2.40)$$

Its controllability grammian is the solution of

$$A^{-1}W_c + W_c A^{-T} + A^{-1}BB^T A^{-T} = 0 \quad (2.41)$$

which is the same as Eq. (2.2). By using A^{-1} as the iteration matrix and $A^{-1}B$ as the starting block of vectors, the proposed Lanczos algorithm will generate a set of Lanczos vectors that transforms the system equation in Eq. (2.40) into the form

$$\begin{aligned} T\dot{\bar{x}} &= \bar{x} + \bar{B}u \\ y &= \bar{C}\bar{x} \end{aligned} \quad (2.42)$$

where T is the block tridiagonal matrix defined in Eq. (2.20). The reduced system in Eq. (2.42) matches the low-frequency moments of the full-order system. If preservation of some Markov parameters are required, $A^k B$ instead of $A^{-1} B$ should be used as the starting block of vectors.

2.2.5 An Observability Grammian Approach

Although the formulation so far has been based on the controllability grammian, there is a dual approach that is based on the observability grammian. The observability grammian is the solution of

$$A^T W_o + W_o A + C^T C = 0 \quad (2.43)$$

If A is replaced by A^T , B is replaced C^T , and W_c is replaced by W_o in the algorithm in Table 2.1.3, then the Lanczos vectors generated lie in the span of

$$\begin{bmatrix} C^T & A^T C^T & \dots & (A^T)^{k-1} C^T \end{bmatrix}$$

which is the *observability matrix* of the system. Hence, the Lanczos vectors generated by the observability grammian approach are in the observable subspace.

The set of Lanczos vectors generated by the observability grammian approach forms an L matrix that satisfies

$$L^T W_o^{-1} L = I \quad , \quad L^T W_o^{-1} A^T L = T$$

Therefore, we can let

$$x = W_o^{-1} L \bar{x}$$

and transform the system equation, Eq. (2.1), to the form

$$\begin{aligned}\dot{\bar{x}} &= T^T \bar{x} + L^T B u \\ y &= C W_o^{-1} L \bar{x}\end{aligned}\tag{2.44}$$

where the transformed output distribution matrix $C W_o^{-1} L$ has nonzero elements only in the first block.

In application to model reduction, the controllability grammian approach can be used to delete the uncontrollable and nearly uncontrollable part of the system. Subsequently, the observability grammian approach can be used to delete the unobservable and nearly unobservable part of the system.

2.3 Numerical Examples

2.3.1 General Linear System Example

The first example is taken from Ref. [3]. This example is used to illustrate the fact that the proposed algorithm can generate a set of Lanczos vectors that are in the span of the controllable and observable subspace. The system is a five-state, unstable, non-minimal system whose system matrices are

$$A = \begin{bmatrix} -3 & -2 & -2 & -2 & -2 \\ -3 & -8 & -11 & -11 & -11 \\ 6 & 12 & 17 & 16 & 16 \\ -3 & -6 & -9 & -9 & -11 \\ 1 & 2 & 3 & 4 & 6 \end{bmatrix} \quad B = \begin{bmatrix} -7.5 \\ -6.0 \\ 10.5 \\ -3.0 \\ 0.0 \end{bmatrix} \quad C^T = \begin{bmatrix} -6.0 \\ -12.0 \\ -16.0 \\ -17.0 \\ -18.0 \end{bmatrix}$$

The eigenvalues of the A matrix are: $-2, -1, 1, 2, 3$. Only two states are both controllable and observable.

Because the system is not stable, we first define a shifted system with system matrix $A_\sigma = A - 5I$. Then, the controllability grammian approach is

applied to the shifted system (A_σ, B, C) with B chosen as the starting vector. The Lanczos algorithm stops at the third iteration. Three Lanczos vectors are generated, and they form a matrix $L_c \in R^{5 \times 3}$, which spans the controllable subspace. Using L_c as the model reduction basis, the following reduced system matrices are obtained

$$A_c = L_c^T W_c^+ A L_c = \begin{bmatrix} -10.0000 & -7.8994 & 0 \\ 7.8994 & 5.0000 & -2.3664 \\ 0 & 2.3664 & 5.0000 \end{bmatrix}$$

$$B_c = L_c^T W_c^+ B = \begin{Bmatrix} 5.4772 \\ 0 \\ 0 \end{Bmatrix} \quad C_c = C L_c = \begin{bmatrix} 0 & -0.8321 & -2.4612 \end{bmatrix}$$

which represent a completely controllable system. However, this reduced system is not a minimal system yet, because it is not completely observable. The eigenvalues of the reduced system are: $-2, -1, 3$.

Next, we can apply the observability grammian approach to the three-state reduced system obtained above to delete the unobservable state(s). Again, because the reduced system is not stable, we define a shifted system matrix $A_r = A_c - 5I$ and apply the Lanczos algorithm to the shifted system. The Lanczos algorithm stops at the second iteration and creates two Lanczos vectors. These two Lanczos vectors form a matrix L_{co} , which spans the controllable and observable subspace. The reduced system matrices based on L_{co} are

$$\bar{A} = L_{co}^T A_c W_o^+ L_{co} = \begin{bmatrix} -3.0000 & 3.4641 \\ -3.4641 & 5.0000 \end{bmatrix}$$

$$\bar{B} = L_{co}^T B_c = \begin{Bmatrix} 0 \\ -2.5981 \end{Bmatrix} \quad \bar{C} = C_c W_o^+ L_{co} = \begin{bmatrix} 4.000 & 0 \end{bmatrix}$$

This is a minimal-order representation of the original five-state system.

Although the modified two-sided nonsymmetric Lanczos algorithm in Ref. [3] also produces a minimal-order model for the same system, the reduced system matrix does not have an almost skew-symmetric, block-tridiagonal form like the one we obtained here. Besides that, the algorithm in Ref. [3] is a single-vector algorithm and, therefore, can be used only for single-input/single-output systems, while the one-sided, unsymmetric Lanczos algorithm proposed here is derived for general multi-input/multi-output systems.

2.3.2 A Flexible Structure Example

The second example is a plane truss structure as shown in Fig. 2.1. This

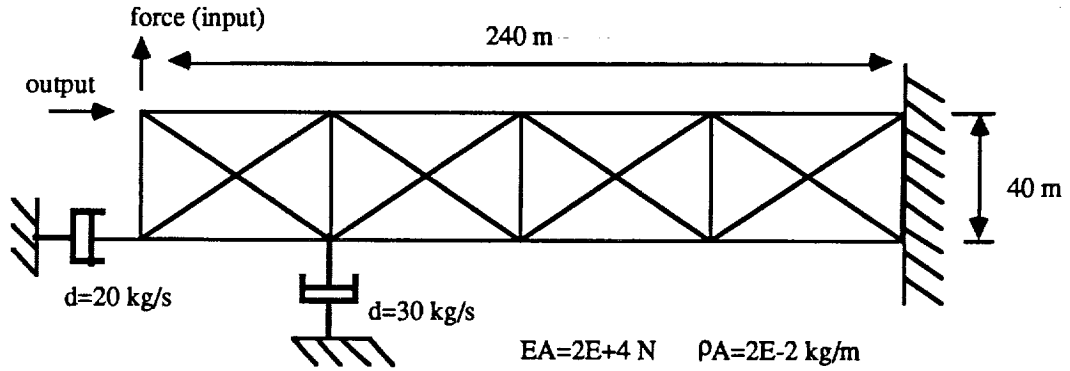


Figure 2.1: A plane truss structure.

plane truss structure has sixteen degrees of freedom, and hence, thirty-two states. There is one force actuator and one displacement sensor on the structure. Because of the dashpot dampers, the damping matrix is non-proportional. The eigenvalues of the system are listed in Table 2.2, which shows that, except for the first mode, which is overdamped, all the modes are underdamped with

damping ratios that range from 0.3% to 9%.

Four reduced-order models are examined: a twelve-state reduced-model based on the twelve complex eigenvectors corresponding to the six lowest frequency modes, and three twelve-state Lanczos reduced-models based on different starting vectors. The Lanczos vectors are generated by using A^{-1} as the iteration matrix. Therefore, the Lanczos-reduced models tend to approximate the low-frequency range of the system. The three starting vectors considered are: $A^{-1}B$, B , and AB .

Figures 2.2 through 2.5 compare the frequency response functions of the reduced-models with that of the full-order model. In the low-frequency range, the Lanczos-reduced models approximate the system equally well as the eigenvector reduced-model. However, in the high-frequency range, the system is much better approximated by the three Lanczos-reduced models than by the eigenvector-reduced model. It is also seen that the Lanczos-reduced model obtained by using AB as the starting vector matches the two Markov parameters CB and CAB , which represent the high frequency behavior of the system, and therefore, it provides the best approximation of system in the high-frequency range. Similar results can be observed from the comparison of impulse response histories in Figs. 2.6 through 2.9.

Table 2.2: Eigenvalues of the plane truss structure.

Eigenvalues ($\times 10^2$)	Damping Ratio
$-0.0324 \pm 0.0233i$	1.3912
$-0.0029 \pm 0.1852i$	0.0155
$-0.0171 \pm 0.1866i$	0.0916
$-0.0301 \pm 0.4134i$	0.0729
$-0.0146 \pm 0.5804i$	0.0251
$-0.0300 \pm 0.6262i$	0.0479
$-0.0087 \pm 0.7836i$	0.0112
$-0.0223 \pm 0.9018i$	0.0247
$-0.0773 \pm 1.0054i$	0.0769
$-0.0031 \pm 1.0087i$	0.0031
$-0.0118 \pm 1.0378i$	0.0114
$-0.0107 \pm 1.3905i$	0.0077
$-0.0332 \pm 1.7021i$	0.0195
$-0.1247 \pm 1.7517i$	0.0712
$-0.0216 \pm 2.0071i$	0.0108
$-0.0499 \pm 2.3059i$	0.0216

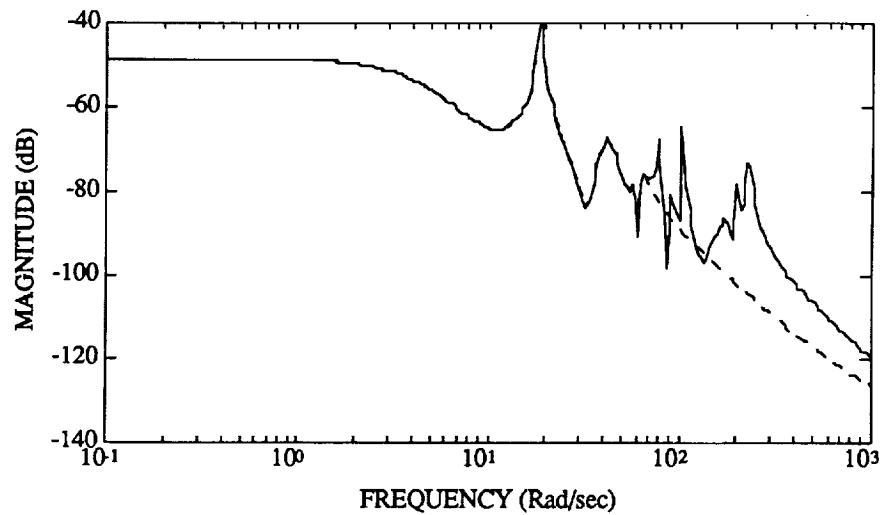


Figure 2.2: Comparison of FRF: full-order model vs twelve-state eigenvector-reduced model.

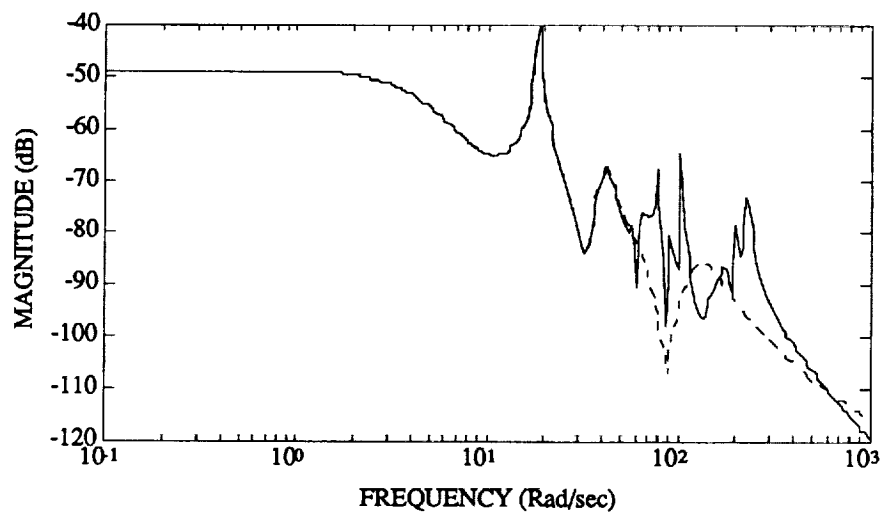


Figure 2.3: Comparison of FRF: full-order model vs twelve-state Lanczos-reduced model with starting vectors $A^{-1}B$.

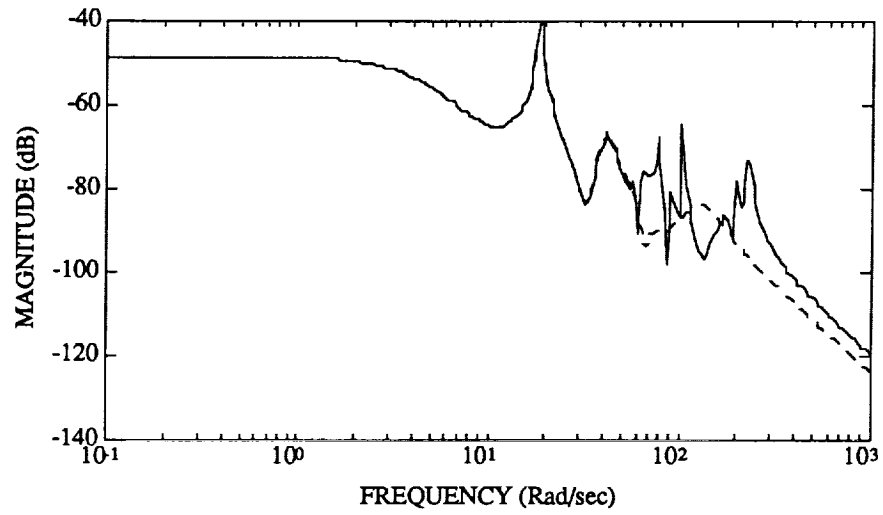


Figure 2.4: Comparison of FRF: full-order model vs twelve-state Lanczos-reduced model with starting vectors B .

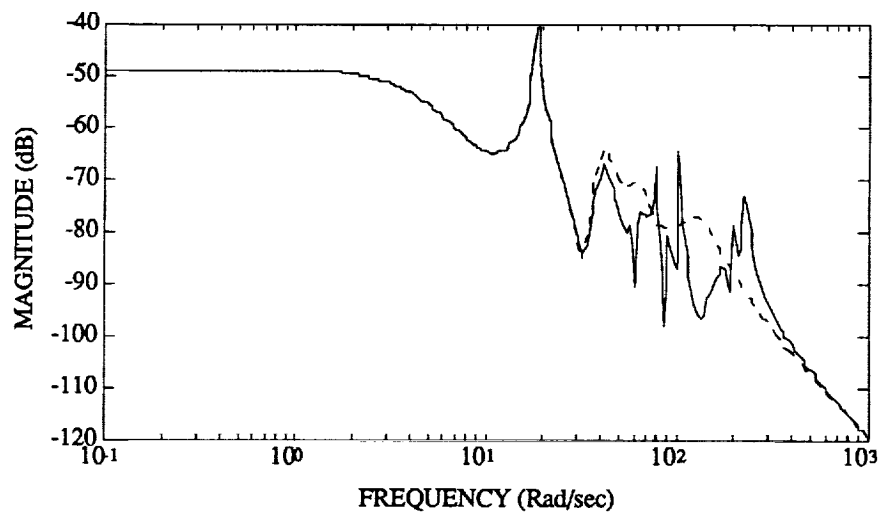


Figure 2.5: Comparison of FRF: full-order model vs twelve-state Lanczos-reduced model with starting vectors AB .

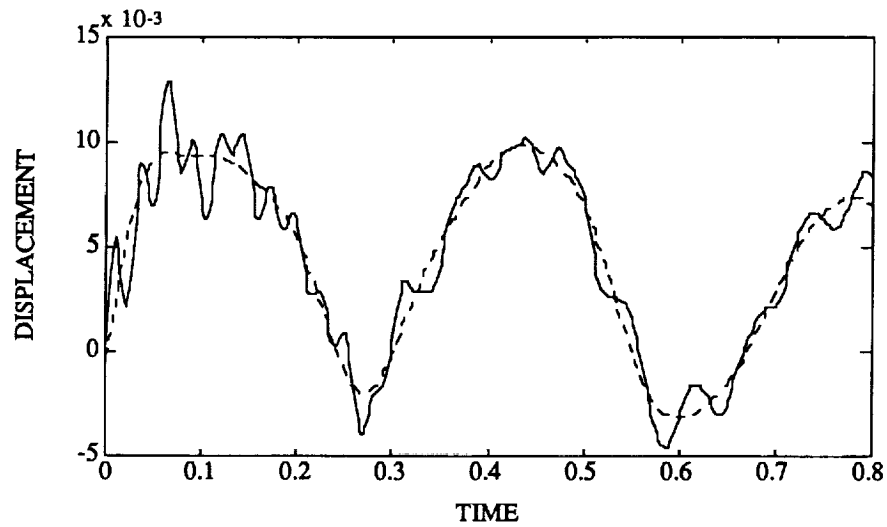


Figure 2.6: Comparison of impulse responses: full-order model vs twelve-state eigenvector-reduced model.

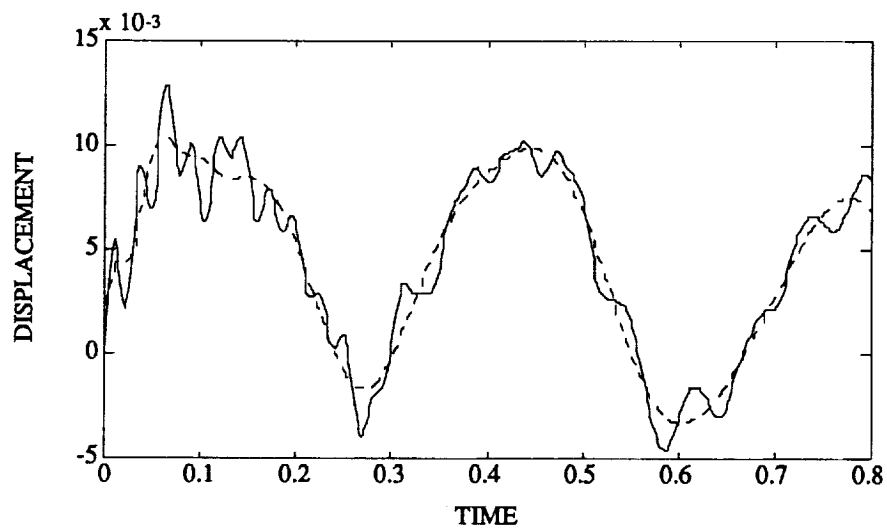


Figure 2.7: Comparison of impulse responses: full-order model vs twelve-state Lanczos-reduced model with starting vectors $A^{-1}B$.

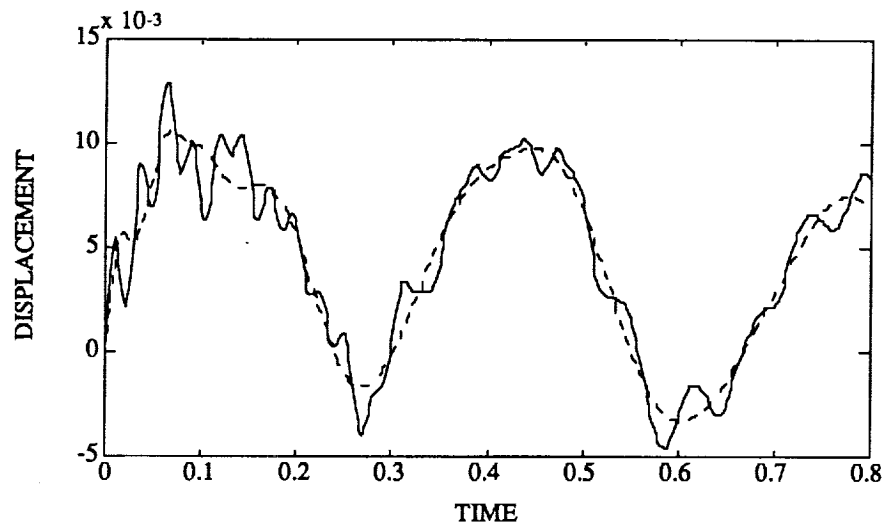


Figure 2.8: Comparison of impulse responses: full-order model vs twelve-state Lanczos-reduced model with starting vectors B .

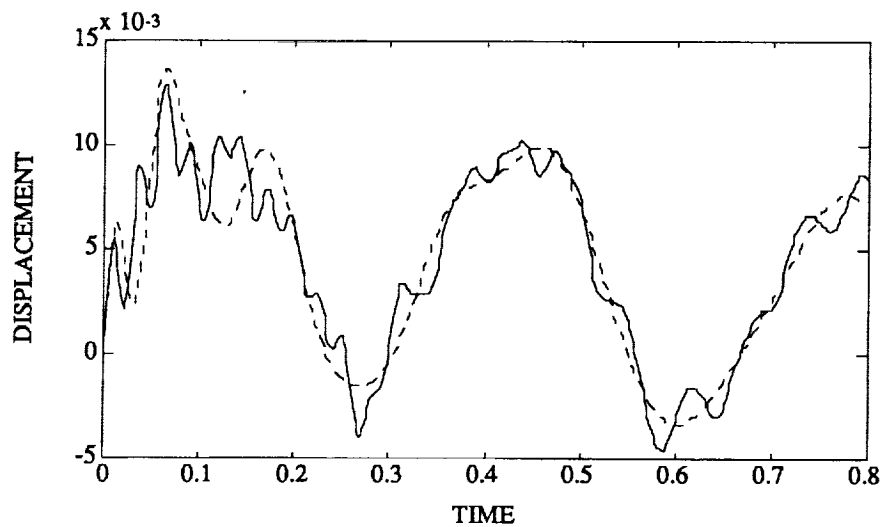


Figure 2.9: Comparison of impulse responses: full-order model vs twelve-state Lanczos-reduced model with starting vectors AB .

Chapter 3

CONTROL DESIGN BASED ON LINEAR STATE FUNCTION OBSERVER

This chapter is concerned with the design of low-order observer-based controllers for large scale systems. Basically, there are three approaches to the design of low-order controllers for large scale systems: model reduction, controller reduction, and parameter optimization. In the model-reduction approach, the size of the system is reduced first, then the design of the controller is based on the reduced model. In the controller-reduction approach, a full-order controller is designed first, then the controller is reduced to a desirable size. In the direct parameter optimization approach, an optimization problem is solved to determine the controller system parameters such that a prescribed performance function is minimized. The first two approaches cause spillover of control energy in the closed-loop system and may take a number of trial and error iterations to reach an acceptable design. The third approach, although theoretically optimal, is numerically intractable for large scale systems.

In this chapter, a new approach to the design of low-order controllers for large scale systems is proposed. The method is derived from the theory of linear state function observers. First, the realization of a state feedback control law is interpreted as the observation of a linear function of the state vector. Observation of linear state functions is an interesting, and yet not completely resolved, topic in observer theory. Luenberger[14] first explored the

possibility of using a low-order observer to reconstruct a single linear state function for single-output systems. For multi-output cases, the first significant contribution was due to Fortmann and Williamson[7]. A complete review and investigation of the reconstruction of linear state functions can be found in the book by O'Reilly[20]. It is shown that there are three conditions that must be satisfied by the observer system matrices. In general, it is not easy, usually not possible, to find an exact observer that satisfies these conditions. Therefore, the method proposed here seeks to design an approximate observer that will produce feedback signals close to those due to the given control law, rather than to design an observer that can realize the given control law exactly.

The observer system is characterized by the damping factors and frequencies of the poles and the elements of the filter gain matrix. These parameters are determined by solving an optimization problem, which yields a matrix that is close to the given control feedback gain matrix. Then, that matrix is used to determine a new control law.

The organization of this chapter is as follows. In Section 3.1, the theory of linear state function observers is reviewed. The conditions that a linear state function observer must satisfy are derived and summarized. In Section 3.2, the formulation of the optimization problem for a control design procedure based on linear state function observers is presented. Section 3.3 uses a four-disk system and a lightly-damped beam as examples to demonstrate the applicability and efficacy of the proposed method. Part of the material in this chapter is presented in Refs. [26] and [27].

3.1 Review of Linear State Function Observers

Consider a linear, time-invariant system of the form

$$\begin{aligned} \dot{x} &= Ax + Bu, & x &\in R^n, u \in R^l \\ y &= Cx, & y &\in R^m \end{aligned} \quad (3.1)$$

and, without loss of generality, assume that (A, B) is completely controllable, (A, C) is completely observable, and that B and C are of full rank. A linear state feedback control law for the above system has the form

$$u = Kx \quad (3.2)$$

where the $l \times n$ feedback gain matrix K is designed to achieve some specified performance objective by using an existing control design approach, for instance, linear quadratic control theory or the eigenvalues/eigenvectors assignment method. Due to the limited output measurement, $y = Cx$, the full state feedback control law in Eq. (3.2) in general cannot be realized (except for the special case with $K = RC$ for some R , which is the case of direct output feedback). In order to implement the state feedback control, an observer is required to reconstruct the complete state vector. It is well known that for an n -th order system with m outputs, it is always possible to construct a reduced-order observer of order $n - m$, with poles arbitrarily placed (subject to complex pairing), to yield an asymptotic estimate of the states. But, for large scale dynamic systems with very few outputs, an observer of order $n - m$ is usually too large for implementation purposes. Hence, it is desirable to design an observer with considerably reduced order r , $r < (n - m)$.

In actual implementation, the control law in Eq. (3.2) does not necessarily require estimation of the complete state vector x . The only required

feedback signal is Kx , which can be considered as a linear function of the state vector. If there exists a low-order observer that estimates Kx , then it can provide the same feedback signal as the full-order state feedback. Since $Kx \in R^{l \times 1}$, where $l \ll n$, it is reasonable to expect that the reconstruction of the linear state function Kx can be accomplished by an observer of order smaller than $n - m$.

In general, it is rarely possible to find a reduced-order observer of dimension l that can reconstruct Kx exactly. So, instead, assume that the feedback gain matrix K can be decomposed into the form

$$K = GT, \quad G \in R^{l \times r}, \quad T \in R^{r \times n} \quad (3.3)$$

where $l < r < n - m$. Then, the feedback control law in Eq. (3.2) becomes

$$u = GTx \quad (3.4)$$

which can be implemented by using the measurement of the linear state function Tx with G as the gain matrix. Now, consider a r -th order observer of the form

$$\dot{q} = Eq + TBu + Fy, \quad q \in R^r \quad (3.5)$$

The observer state vector q is to approximate Tx in the following asymptotic sense

$$\lim_{t \rightarrow \infty} [q(t) - Tx(t)] = 0 \quad (3.6)$$

It is apparent that the observer system matrices E and F cannot be arbitrary and that they must satisfy some condition(s). Define the error vector

$$\varepsilon = q - Tx \quad (3.7)$$

Then, from Eqs. (3.1) and (3.5),

$$\dot{\varepsilon} = E\varepsilon + FCx - TA\varepsilon \quad (3.8)$$

If E and F are chosen to satisfy the Lyapunov equation

$$ET - TA + FC = 0 \quad (3.9)$$

then Eq. (3.8) becomes

$$\dot{\varepsilon} = E\varepsilon \quad (3.10)$$

The error dynamics is governed by the stability of the E matrix.

In summary, in order to realize the control law in Eq. (3.2) exactly, the observer described by Eq. (3.5) must satisfy the following three conditions:

- (i) E is a stable matrix,
- (ii) $ET - TA + FC = 0$, and
- (iii) $K = GT$

Also, for Eq. (3.9) to possess a unique solution for T , the E and A matrices cannot have common eigenvalues[15].

Now, suppose that the observer in Eq. (3.5) satisfies the estimation requirement. Then, the control signal can be set to be $u = Gq$, which leads to the following closed-loop system equation

$$\begin{Bmatrix} \dot{x} \\ \dot{q} \end{Bmatrix} = \begin{bmatrix} A & BG \\ FC & E + TBG \end{bmatrix} \begin{Bmatrix} x \\ q \end{Bmatrix} \quad (3.11)$$

Or, by using the definition of the error vector ε , the above equation can be

transformed into the following block triangular form

$$\begin{Bmatrix} \dot{x} \\ \dot{\epsilon} \end{Bmatrix} = \begin{bmatrix} A + BGT & BG \\ 0 & E \end{bmatrix} \begin{Bmatrix} x \\ \epsilon \end{Bmatrix} \quad (3.12)$$

This expression indicates that the observer dynamics and controller dynamics are decoupled. The closed-loop poles are the eigenvalues of $A + BGT$ and E . Therefore, controller design and observer design can be separated in the same manner as in the design of a full-order observer.

Sufficient and necessary conditions for the existence of the E , F , and T matrices, and an efficient numerical scheme for the their solution are still not available. In fact, by examining the number of equations ($r \times n$) and the number of unknowns ($(r \times m) + (r \times r)$, assuming T given) in Eq. (3.9), it is concluded that for the cases with r and m much smaller than n , in general the problem is overdetermined and a solution does not exist. Therefore, it will not be attempted here to establish a solution scheme for finding an exact reduced-order observer for linear state functions. In the following section, the three conditions for a valid linear state function observer will be used to develop a practical procedure to design low-order observers.

3.2 A Semi-Inverse Observer-Based Control Design Procedure

In the full-order LQR design, the feedback control law is derived from optimal control theory, and the observer is designed by pole placement with the observer poles placed such that the observer dynamics is faster than the dynamics of the rest of the closed-loop system. It is just natural to design the

optimal control law first and the observer afterwards. However, the performance of an optimal regulator is usually degraded when the feedback signal is based on the estimated states instead of the exact states. Newmann[16] showed that there is no way of defining an observer system so that the performance cost will be minimized. Therefore, control design and observer design basically can be considered as two independent processes. A completely inverse procedure whereby the observer is designed first, followed by the determination of the control gain matrix is also adequate, as long as the closed-loop system meets some prescribed performance criteria.

From the above argument, a possible inverse design procedure for a linear state feedback control law and its associated linear state function observer may be described by the following steps:

- (1) Assign matrices E and F for the observer equation, Eq. (3.5).
- (2) Solve the Lyapunov equation, Eq. (3.9), for the T matrix.
- (3) Determine a control law $u = GTx$ with G such that $A + BGT$ is a stable matrix.

It is obvious that if a control law is successfully determined by the above procedure, then the closed-loop poles are the eigenvalues of E and $A + BGT$. The above inverse design procedure, although it may be feasible, is not really practical. First of all, the choice of E and F matrices is arbitrary and there is no criterion on which to base their selection (except that E must be a stable matrix and must not have any eigenvalue in common with A). Secondly, determination of the G matrix can be very difficult. In fact, it is the same problem as the

determination of an output feedback gain for a system with output vector $y = Tx$. A general condition of pole assignability given by Srinathkumar[21] indicates that if (A, B) is completely controllable and (A, T) is completely observable, then $\max(l, r)$ eigenvalues of $A + BGT$ can be arbitrarily specified (subject to complex paring) to within any degree of accuracy. However, the locations of the rest of the eigenvalues are not predictable, and it is never an easy task to determine a G matrix such that all the poles of $A + BGT$ are stable, especially for the case of r much smaller than n . Therefore, a completely inverse design procedure is not practical.

If a feedback gain matrix K is already designed and given, it is reasonable to choose the E and F matrices in such a way that the T matrix obtained from the Lyapunov equation, Eq. (3.9), is “close” to K . The closeness of two matrices can be defined by the following two-norm measure

$$\gamma = \frac{\|K - GT\|_2}{\|K\|_2} \quad (3.13)$$

If T is close to K , which means that the subspace spanned by the row vectors in T is close to the subspace spanned by the row vectors in K , then there exists a G matrix such that γ is small. The G matrix that minimizes γ is

$$G = KT^+ \quad (3.14)$$

where T^+ is the Moore-Penrose pseudo-inverse of T . If T is close enough to K , the closed-loop matrix $A + BGT$ will be stable, providing that $A + BK$ is a stable matrix.

To determine E and F matrices so that they yield a T matrix as close to K as possible, a parameter optimization technique can be used. All the

elements of E and F are considered as optimization parameters of the following optimization problem:

$$\text{Minimize } \gamma(E, F) = \frac{\|K - KT^+T\|_2}{\|K\|_2}$$

Subject to

$$ET - TA + FC = 0$$

E is a stable matrix

This optimization approach, however, has disadvantages in that it involves a large number of parameters, and the order of the observer needs to be specified first before the optimization can be carried out. Therefore, instead of solving the global optimization problem, a block-by-block optimization procedure will be pursued.

Although $E \in R^{r \times r}$ and $F \in R^{r \times m}$, only $r \times (m + 1)$ parameters are needed to characterize the observer system. The form of the E matrix can be chosen to be block-diagonal

$$E = \text{diag}(\Lambda_i) \quad (3.15)$$

in which

$$\Lambda_i = \begin{bmatrix} -\sigma_i & -\omega_i \\ \omega_i & -\sigma_i \end{bmatrix} \quad (3.16)$$

is the i -th block associated with the pair of complex conjugate poles $-\sigma_i \pm j\omega_i$. For first-order poles, the E matrix has negative real numbers on the diagonal. With the E matrix chosen in the block-diagonal form, the Lyapunov equation (3.9) can be split into independent partitions as

$$\begin{bmatrix} \Lambda_1 & & \\ & \Lambda_2 & \\ & & \ddots \end{bmatrix} \begin{bmatrix} t_1 \\ t_2 \\ \vdots \end{bmatrix} - \begin{bmatrix} t_1 \\ t_2 \\ \vdots \end{bmatrix} A + \begin{bmatrix} f_1 \\ f_2 \\ \vdots \end{bmatrix} C = 0 \quad (3.17)$$

or,

$$\Lambda_i t_i - t_i A + f_i C = 0, \quad i = 1, 2, \dots \quad (3.18)$$

where t_i and f_i are the i -th partitions of T and F corresponding to the i -th block of E . The optimization parameters in Eq. (3.18) are σ_i , ω_i , and the elements of f_i . If it is necessary to characterize damping directly, the damping factor $\zeta_i = \sigma_i/\omega_i$ can replace σ_i as one of the optimization parameters.

A block-by-block optimization procedure is established based on Eq. (3.17). For the i -th partition, the optimization problem is described as:

$$\left. \begin{aligned} &\text{Minimize } \gamma(\zeta_i, \omega_i, f_i) = \frac{\|K - K T_i^+ T_i\|_2}{\|K\|_2} \\ &\text{where } T_i = \begin{bmatrix} T_{i-1} \\ t_i \end{bmatrix}, \text{ with } t_i \text{ being the solution of} \\ &\quad t_i A = \Lambda_i t_i + f_i C \\ &\text{and subject to} \\ &\quad \zeta_{i_{\min}} \leq \zeta_i \leq \zeta_{i_{\max}}, \quad \omega_{i_{\min}} \leq \omega_i \leq \omega_{i_{\max}} \end{aligned} \right\} \quad (3.19)$$

Starting out with $T_0 = [0]$, the observer system matrices E and F are optimized block by block with the objective of obtaining a T matrix that is close to K . Although block-by-block optimization does not yield a global minimum, it has fewer optimization parameters and the computation cost is more economic than a global optimization approach. A general criterion for choosing the bounds for ζ_i and ω_i is to place the observer poles such that the observation process is fast enough to provide estimated states for feedback.

Based upon the above derivation, a semi-inverse observer-based control design procedure can be summarized by the following steps:

- (1) Determine a state feedback gain matrix K . Set $i = 1$, and $T = [0]$.
- (2) Set the bounds for ζ_i and ω_i . Solve the optimization problem (3.19) for t_i , Λ_i , and f_i .
- (3) Expand the T matrix by adding in t_i . Check the stability of $A + BGT$ ($G = KT^+$).
- (4) Determine if the closed-loop system is stable with acceptable performance. If yes, stop; else set $i = i + 1$, and go to Step (2).

The above control design procedure is called a semi-inverse procedure because the final control law is not designed before the observer, but is a result obtained from the observer design. The observer design, however, is not completely independent of the control design, because it tries to yield a control law that is close to a given design. The dimension of the observer does not have to be specified in advance. Hence, the proposed method offers considerable flexibility for the design of a linear state feedback control law and its associated linear state function observer. As the T matrix expands in Step (3), the order of the observer grows. If, at certain point the stability property and the performance of the closed-loop system is satisfactory, the design procedure can stop there. The observer that is obtained has a dimension that is the same as the number of row vectors in the T matrix.

The optimization problem, Eq. (3.19), can be solved numerically by using available efficient optimization schemes. For the example to be demonstrated

later in this paper, a Newton-Raphson method with a one-dimensional search scheme was used. The most time-consuming computation in the optimization process is solving the Lyapunov equation, Eq. (3.18). The general approach to solve the Lyapunov equation is to first transform the A and Λ_i matrices into a simple form, e.g., the upper quasi-triangular form[1], and then to solve the transformed equation. Although the Lyapunov equation is to be solved repeatedly during the optimization process, the transformation of the A matrix needs to be done only once.

The stability of the control design obtained from the semi-inverse design procedure is determined by the location of the regulator poles, which are the eigenvalues of $A + BGT$. For the original design with the feedback gain matrix K , the regulator poles are the eigenvalues of $A + BK$. Since GT approximates K , the eigenvalues of $A + BGT$ can be considered as perturbations of eigenvalues of $A + BK$. Use the following expression:

$$A + BGT = A + BK + B(GT - K) = (A + BK) + B\Delta K \quad (3.20)$$

where

$$\Delta K = GT - K \quad (3.21)$$

is the difference between the new feedback gain matrix and the original gain matrix, which is minimized in the optimization process. If the original design $A + BK$ satisfies stability and performance requirements, the new design will likely be stable if the perturbation term $B\Delta K$ is small. In fact, eigenvalue sensitivity of a matrix due to perturbation is governed by the condition number of the matrix form by the eigenvectors[4]. If the conditioning is bad, even small perturbation can lead to significant changes of eigenvalues. Therefore, it

is recommended that some existing robust eigensystem assignment algorithm, e.g., the method of Kautsky et al.[9], be used to design the state feedback gain matrix K , such that the eigenvalues of $A + BK$ will be insensitive to the system uncertainties and/or perturbations.

3.3 Numerical Examples

3.3.1 A Four Disk System

The first example is a four disk system whose system matrices are listed in Table 3.1. They represent a linear, time-invariant, SISO, unstable and non-

Table 3.1: Data matrices of a four disk system.

$A =$	$\begin{bmatrix} -0.1610 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -6.0040 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ -0.5822 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ -9.9835 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ -0.4073 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ -3.9820 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$
$B^T =$	$[0, 0, .0064, .00235, .0713, 1.0002, .1045, .9955]$
$C =$	$[1 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0]$
$Q =$	$(1.0 \times 10^{-6})H^T H;$
$H =$	$[0 \ 0 \ 0 \ 0 \ 0.55 \ 11 \ 1.32 \ 18.0]$
$R =$	1

minimum phase system of order eight. This example has been used to compare different controller reduction methods[13, 25]. It will be used here to demon-

strate the feasibility and efficacy of the proposed method.

The original feedback gain matrix K was determined by using the LQR control design method. Then, three observer designs were obtained by using the semi-inverse control design procedure. The results are summarized in Table 3.2, and Figs. 3.1 through 3.3. The parameter bounds were set to be $\omega \geq 0.01$ and

Table 3.2: Three observer designs.

Design	Order	Poles	γ	Cost [†]
1	2	$-.1099 \pm j.0275$.0220	.1708
2	3	$-.1099 \pm j.0275,$ -0.9	.0180	.1608
3	4	$-.1099 \pm j.0275,$ $-.3554 \pm j.0888$.0133	.1605

† With exact state feedback, the minimum cost is 0.1591.

$\zeta \geq 2$, which means that the observer is an overdamped system. Design 1 is a second-order linear state function observer, which is the result of optimization for the first block of the E matrix. The values of γ and the performance cost indicate that the control law of Design 1 is very close to that of the optimal design. The original gain matrix and the gain matrix obtained from the semi-inverse design procedure are

$$K = [0.0001 \ 0.0013 \ 0.0002 \ 0.0010 \ -0.0009 \ -0.0032 \ 0.0060 \ 0.1046]$$

$$GT = [0.0000 \ 0.0000 \ 0.0000 \ -0.0001 \ 0.0005 \ -0.0027 \ 0.0060 \ 0.1046]$$

It is seen that the new gain matrix GT picks up all the largest gain elements in the K matrix. The performance cost of the second-order observer is also

very close to the minimum performance cost, 0.1591, which is calculated by assuming that all of the system states are measurable for feedback.

Design 2 is a third-order linear state function observer, which is obtained by augmenting Design 1 by a first-order pole. Design 3 is a fourth-order linear state function observer with two pairs of complex conjugate poles. All three designs are stable with satisfactory performance. The control law of Design 3 is nearly optimal. Table 3.3 compares the eigenvalues of $A + BGT$ of Design 3

Table 3.3: Comparison of eigenvalues.

$A + BGT$	$A + BK$
$-0.0369 \pm j1.8495$	$-0.0370 \pm j1.8496$
$-0.0284 \pm j1.4100$	$-0.0283 \pm j1.4097$
$-0.0167 \pm j0.7656$	$-0.0167 \pm j0.7652$
$-0.0493 \pm j0.0275$	$-0.0492 \pm j0.0371$

and the eigenvalues of $A + BK$. The condition number of the modal matrix of $A + BK$ is 119.6, which is not considered to be very small. However, the norm of the perturbation term $\|B\Delta K\|_2 = 0.002$ is much smaller than the norm of the original closed-loop matrix $\|A + BK\|_2 = 12.4$. This explains why the perturbations of the regulator poles are small.

Response comparisons are shown in Figs. 3.1 through 3.3. The initial states are arbitrarily set to be $x_0 = [1 \ 1 \ 1 \ 1 \ 1 \ 1 \ 1 \ 1]$. Again, the fourth-order observer tracks the optimal regulation trajectory almost exactly. This example shows that if the optimization can yield a T matrix very close to the original feedback gain matrix, then the control law derived from the linear state

function observer is also close to the prescribed control law.

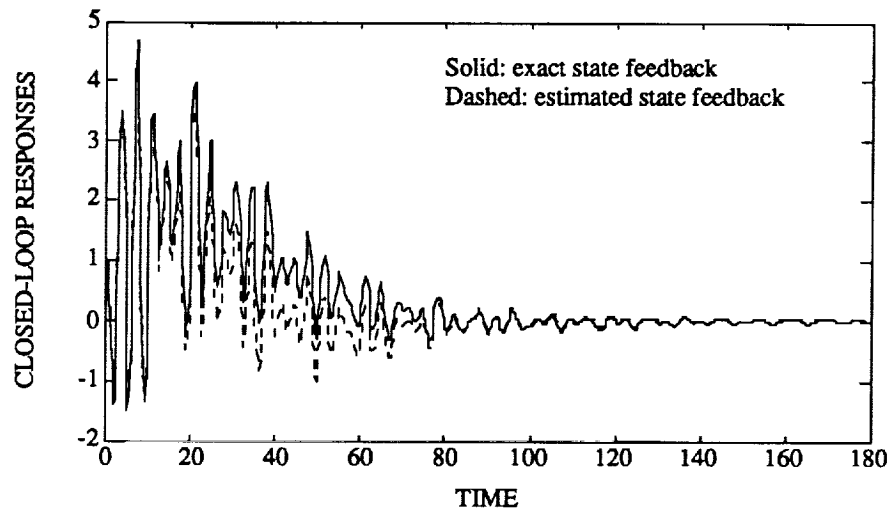


Figure 3.1: Response comparison: Design 1 vs exact state feedback.

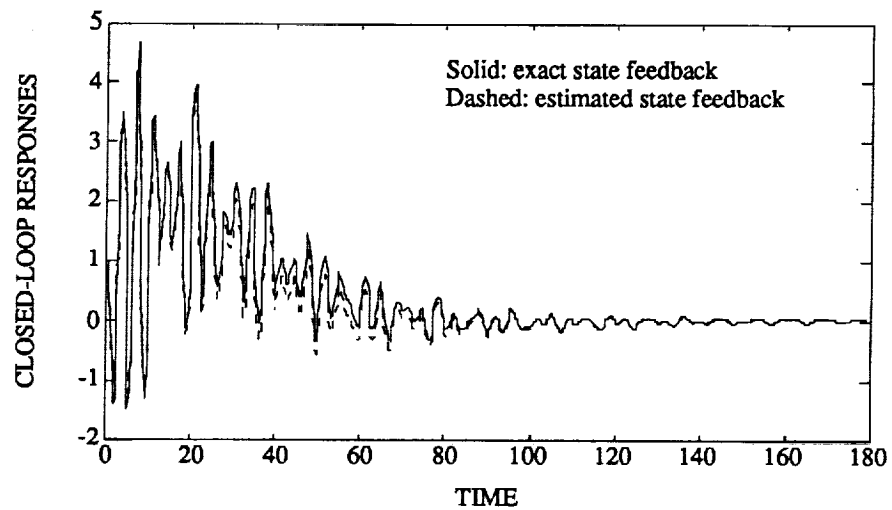


Figure 3.2: Response comparison: Design 2 vs exact state feedback.

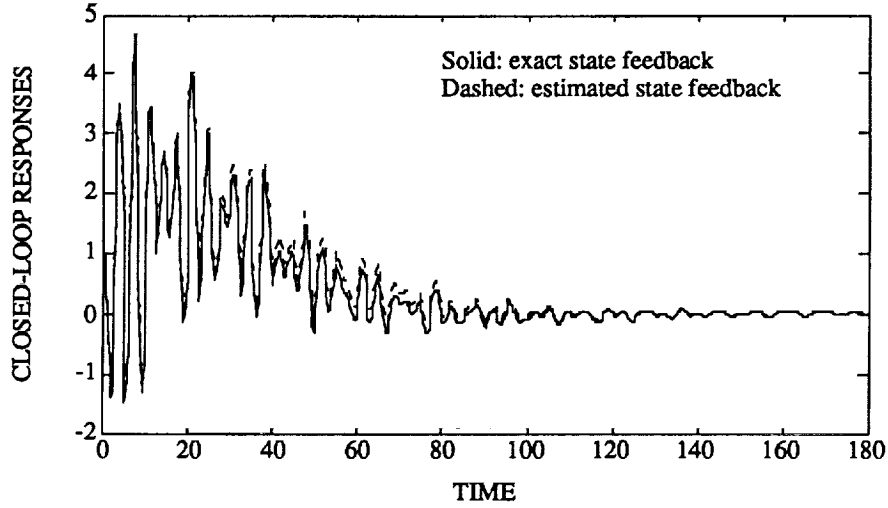


Figure 3.3: Response comparison: Design 3 vs exact state feedback.

3.3.2 A Lightly-Damped Beam

The second example is a simply-supported lightly-damped beam taken from [28]. Table 3.4 shows the data matrices of this system. Two optimal

Table 3.4: Data matrices of a lightly damped beam.

$$A = \text{block diag} \left(\begin{bmatrix} 0 & 1 \\ -\omega_i^2 & -2\zeta\omega_i \end{bmatrix} \right), \quad i = 1, \dots, 5$$

$$B^T = [0, .9877, 0, -.0309, 0, -.8910, 0, -.5878, 0, .7071]$$

$$C = [.9877, 0, .3090, 0, -.8910, 0, -.5878, 0, .7071, 0]$$

$$Q = C^T C;$$

$$R = 100, 0.1$$

feedback gain matrices are determined by using the LQR control design method with control weighting $R = 100$ and $R = 0.1$ respectively. The parameter bounds for optimization are set to be $10 \geq \zeta \geq 0.1$ and $\omega \geq 0.1$.

For the case with low-gain control ($R = 100$), the results are summarized in Fig. 3.4 and Table 3.5. It is seen that as the order of the observer increases,

Table 3.5: Optimization results ($R = 100$).

Order	Block	Poles	γ	Cost [†]
2	Λ_1	$-0.0875 \pm j0.8748$	0.1727	14.7234
4	Λ_2	$-0.2483 \pm j2.4831$	0.1248	13.7059
6	Λ_3	$-0.6332 \pm j2.1718$	0.0035	13.1212

† With exact state feedback, the minimum cost is 13.1200.

the value of γ decreases and the controller performance approaches the optimal one. In fact, the second-order linear state function observer is already a good approximation of the optimal design. The results for the case with high-gain control ($R = 0.1$) are summarized in Table 3.6 and Figs. 3.5 and 3.6. The

Table 3.6: Optimization results ($R = 0.1$).

Order	Block	Poles	γ	Cost [†]
2	Λ_1	$-1.9582 \pm j0.1958$	0.4236	unstable
4	Λ_2	$-0.2023 \pm j2.0233$	0.1397	3.0000
6	Λ_3	$-0.5477 \pm j5.4773$	0.1036	2.4981
8	Λ_4	$-1.2263 \pm j10.000$	0.0747	2.5114

† With exact state feedback, the minimum cost is 2.1673.

second-order linear state function observer did not yield a stable design. The fourth- and sixth-order linear state function observers, however, yield stable

designs with satisfactory performance. Note that in Table 3.6 although the γ value of the eighth-order observer is smaller than that of the sixth-order observer, the cost value of the eighth-order observer is higher than that of the sixth-order observer. This result comes from the fact that optimization process minimizes γ instead of the cost function. According to the definition of closeness, Eq. (3.13), it is still fair to say that the eighth-order observer is closer to the given control law than is the sixth-order observer. The cost values were compared here simply because the given feedback gain matrix is an optimal control law. If the original feedback gain matrix is obtained by using some eigenvalues/eigenvectors assignment algorithm, then comparison of the cost values is not a mandatory criterion.

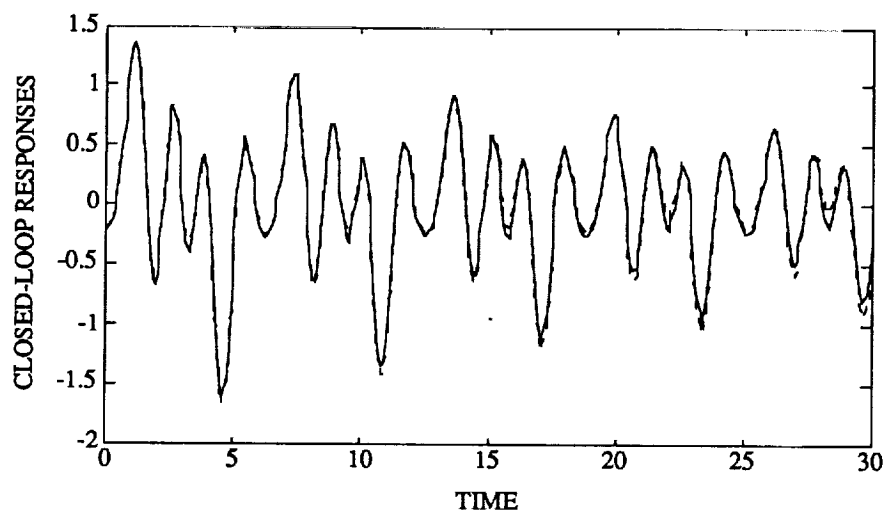


Figure 3.4: Response comparison ($R = 100$): second-order linear state function observer vs exact state feedback.

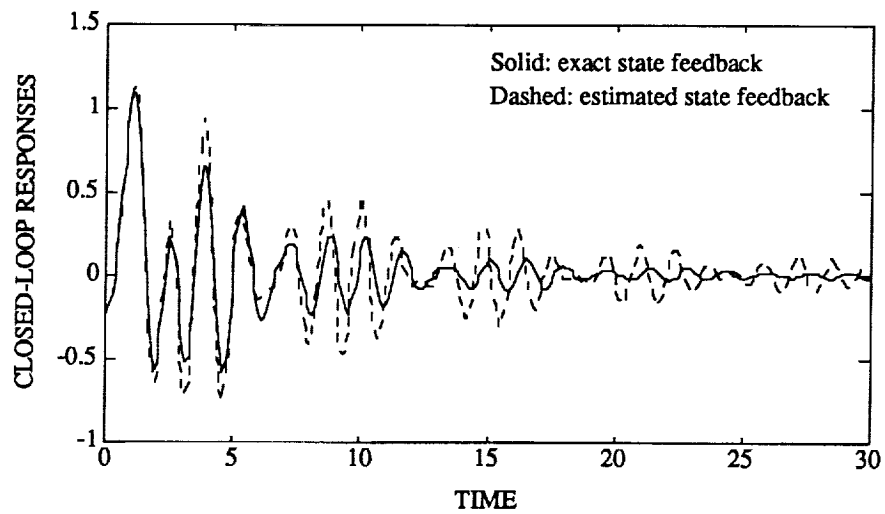


Figure 3.5: Response comparison ($R = 0.1$): fourth-order linear state function observer vs exact state feedback.

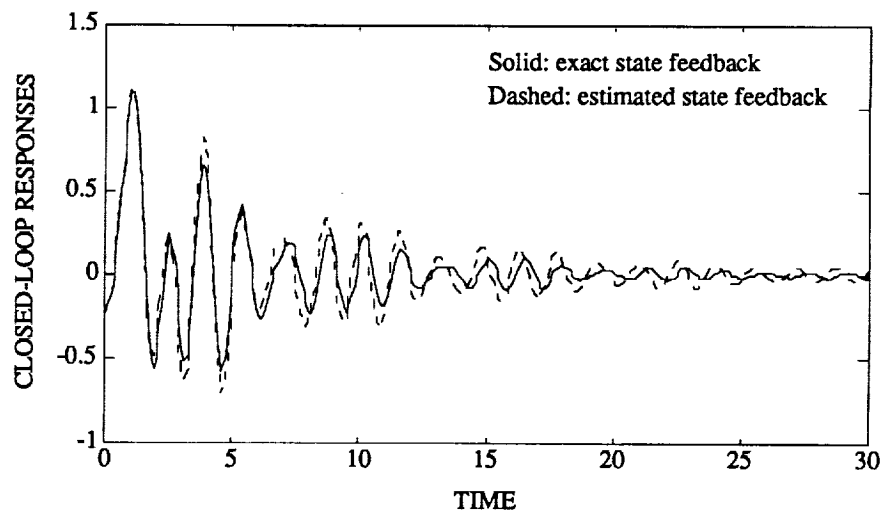


Figure 3.6: Response comparison ($R = 0.1$): sixth-order linear state function observer vs exact state feedback.

Chapter 4

CONCLUSIONS

A one-sided, unsymmetric block Lanczos algorithm for the model reduction of damped structural dynamics systems is presented in this report. The most expensive computation involved in the algorithm is the solution of the Lyapunov equation. There are several advantages of the proposed method over the existing Lanczos algorithm. First, the numerical breakdown problem which usually occurs in the two-sided unsymmetric Lanczos algorithm is not present in the one-sided algorithm. It is also shown that the Lanczos vectors generated by the proposed algorithm lie either in the controllable subspace or the observable subspace, depending on whether the controllability grammian or the observability grammian is used as the normalization weighting matrix. The reduced-order model based on the Lanczos vectors is guaranteed to be stable. For unstable systems, a shifting scheme can be used to ensure the solution of the Lyapunov equation. It is shown that shifting of the system matrix does not change the span of the Lanczos vectors. Finally, the flexibility of the choice of starting vector yields more accurate reduced-order models.

A semi-inverse observer-based control design procedure is also presented in this report. The method offers significant flexibility for the design of low-order controllers. Two examples have demonstrated that the semi-inverse scheme indeed can produce low-order controllers with satisfactory performance.

The proposed method does not require model reduction or controller reduction. Therefore, there is no control energy spillover in the closed-loop system. To yield a very-low-order controller, it is required that the eigenvalues of the original feedback design be insensitive to closed-loop system parameter perturbation. A possible improvement of the proposed procedure would be to approximate the control gain matrix obtained from a robust pole placement method instead of from the LQR control theory. The heaviest computation cost in the optimization process is the solution of the Lyapunov equation. The selection of the parameter bounds is on a trial and error basis. More study should be devoted to the selection of good observer poles.

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